EE 508
Lecture 8

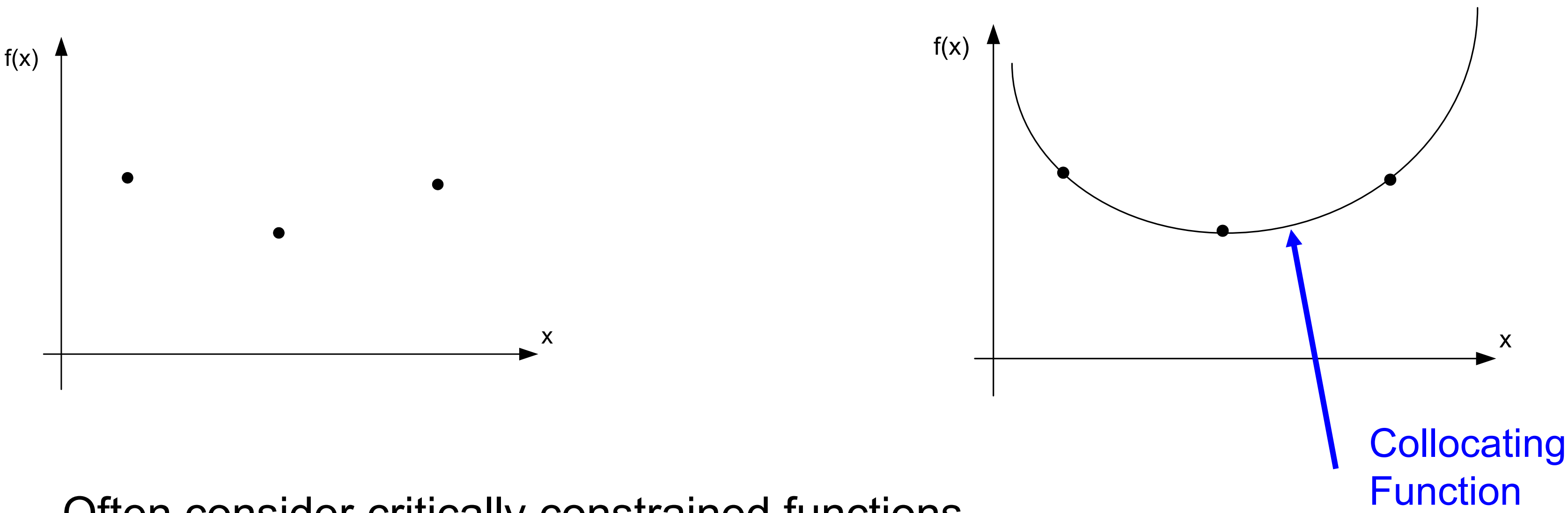
The Approximation Problem

Least squares approximations
Pade Approximations
Numerical Optimization
Classical Approximations

Review from Last Time

Collocation

Collocation is the fitting of a function to a set of points (or measuremetns) so that the functin agrees wth the sample at each point in the set.



Often consider critically constrained functions

The function that is of interest for using collocation when addressing the approximation problem is $H_A(\omega^2)$

Review from Last Time

Collocation

Applying to $H_A\left(\omega^2\right)$

$$\left\{(\omega_1, y_1), (\omega_2, y_2)\ldots(\omega_k, y_k)\right\}$$

$$H_A\left(\omega^2\right) = \frac{a_0 + a_1\omega^2 + a_2\omega^4 + \ldots + a_m\omega^{2m}}{1 + b_1\omega^2 + b_2\omega^4 + \ldots + b_n\omega^{2n}}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \bullet \\ \bullet \\ y_k \end{bmatrix} = \begin{bmatrix} 1 & \omega_1^2 & \omega_1^4 \ldots \omega_1^{2m} & -\omega_1^2 y_1 & -\omega_1^4 y_1 & -\ldots-\omega_1^{2n} y_1 \\ 1 & \omega_2^2 & \omega_2^4 \ldots \omega_2^{2m} & -\omega_2^2 y_1 & -\omega_2^4 y_1 & -\ldots-\omega_2^{2n} y_1 \\ \bullet \\ \bullet \\ 1 & \omega_k^2 & \omega_k^4 \ldots \omega_k^{2m} & -\omega_k^2 y_1 & -\omega_k^4 y_1 & -\ldots-\omega_k^{2n} y_1 \end{bmatrix} \bullet \begin{bmatrix} a_0 \\ a_1 \\ .. \\ a_m \\ b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix}$$

$$\mathbf{Y} = \mathbf{Z} \bullet \mathbf{C}$$

$$\mathbf{C} = \mathbf{Z}^{-1} \bullet \mathbf{Y}$$

Collocation Observations

Fitting an approximating function to a set of data or points (collocation points)

- Closed-form matrix solution for fitting to a rational fraction in ω^2
- Can be useful when somewhat nonstandard approximations are required
- Quite sensitive to collocation points
- Although function is critically constrained, since collocation points are variables, highly under constrained as an optimization approach
- Although fit will be perfect at collocation points, significant deviation can occur close to collocation points
- Inverse mapping to $T_A(s)$ may not exist
- Solution may not exist at specified collocation points

Review from Last Time

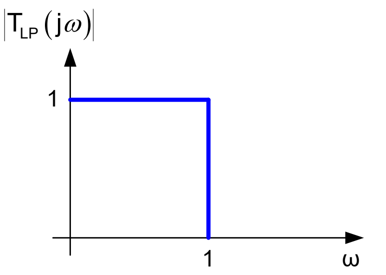
Collocation

What is the major contributor to the limitations observed with the collocation approach?

- Totally dependent upon the value of the desired response at a small but finite set of points   (no consideration for anything else)
- Highly dependent upon value of approximating function at a single point or at a small number of points
- Highly dependent upon which points are chosen

Review from Last Time

The Approximation Problem



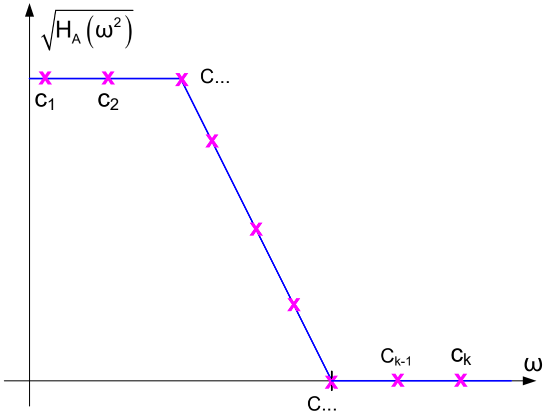
Approach we will follow:

- Magnitude Squared Approximating Functions $H_A(\omega^2)$
- Inverse Transform $H_A(\omega^2) \rightarrow T_A(s)$
- Collocation
- Least Squares (Cost function minimizations)
- Pade Approximations
- Other Analytical Optimization
- Numerical Optimization
- Canonical Approximations
  - → Butterworth (BW)
  - → Chebyschev (CC)
  - → Elliptic
  - → Bessel
  - → Thompson

Review from Last Time

Cost Function Minimizations

$$H_A(\omega^2) = \frac{\sum_{i=0}^{m} a_i \omega^{2i}}{1 + \sum_{i=1}^{n} b_i \omega^{2i}}$$



$$\varepsilon_i = H_D(\omega_i) - H_A(\omega_i)$$

Goal is to minimize some metrics associated with ε_i at a large number of points

Some possible cost functions

$$C_1 = \sum_{i=1}^{N} |\varepsilon_i| \qquad C_2 = \sum_{i=1}^{N} \varepsilon_i^2$$

$$C_3 = \sum_{i=1}^{N} w_i \varepsilon_i^2 \qquad C_{w:m} = \sum_{i=1}^{N} w_i |\varepsilon_i|^m$$

$$C_{w:m_1,m_2} = \sum_{i=1}^{N_1} w_i |\varepsilon_i|^{m_1} + \sum_{i=N_1+1}^{N} w_i |\varepsilon_i|^{m_2}$$

w_i a weighting function

Termed “L_m norm” if exponent is m and weight is 1

- Reduces emphasis on individual points
- Some much better than others from performance viewpoint
- Some much better than others from computation viewpoint
- **Realization of no concern how approximation obtained, only of how good it is !**

Least Squares Approximation

$$H_A(\omega^2) = \frac{\sum_{i=0}^{m} a_i \omega^{2i}}{1 + \sum_{i=1}^{n} b_i \omega^{2i}}$$

$$\varepsilon_i = H_D(\omega_i) - H_A(\omega_i)$$

Consider:

$$C_3 = \sum_{i=1}^{N} w_i \varepsilon_i^2$$

w_i a weighting function

If exponent in cost function is 2, termed "least squares" cost function

Least Mean Square (LMS) based cost functions have minimums that can be analytically determined for some useful classes of approximating functions $H_A(\omega^2)$

- Often termed a L_2 norm
- Minimizing L_1 norm often provides better approximation but no closed-form analytical expressions
- Most of the other metrics listed on previous slide are not easy to get closed-form expressions for minimums though computer optimization can be used: may be plagued by multiple local minimums but they may still be useful

Review from Last Time

Regression Analysis Review

Consider an nth order polynomial in x

$$F(x) = \sum_{k=0}^{n} a_k x^k$$

Consider N samples of a function $\tilde{F}(x)$

$$\hat{F}(x) = \left\langle \tilde{F}(x_i) \right\rangle_{i=1}^{N}$$

where the sampling coordinate variables are

$$X = \langle x_i \rangle_{i=1}^{N}$$

Define the summed square difference cost function as

$$C = \sum_{i=0}^{N} \left(F(x_i) - \tilde{F}(x_i) \right)^2$$

A standard regression analysis can be used to minimize C with respect to $\{a_0, a_1, \ldots a_n\}$

To do this, take the n+1 partials of C wrt the a_i variables

Review from Last Time

Regression Analysis Review

$$C = \sum_{i=0}^{N} \left(F(x_i) - \tilde{F}(x_i) \right)^2 \qquad F(x) = \sum_{k=0}^{n} a_k x^k$$

$$C = \sum_{i=0}^{N} \left(\sum_{k=0}^{n} a_k x_i^k - \tilde{F}(x_i) \right)^2$$

$$A = X^{-1} \bullet F$$

Observations about Regression Analysis:

- Closed form solution
- Requires inversion of a (n+1) dimensional square matrix
- Not highly sensitive to any single measurement
- Widely used for fitting a set of data to a polynomial model
- Points need not be uniformly distributed
- Adding weights does not complicate solution

This analysis was restricted to a polynomial – will see how applicable it is to a rational fraction !

Least Squares Approximations of Transfer Functions

$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i}$$

WLOG $b_0 = 1$

$$T(j\omega) = \frac{\left[\sum_{\substack{i=0 \\ i\ odd}}^{m} (-1)^i a_i \omega^i\right] + \left[\sum_{\substack{i=0 \\ i\ even}}^{m} (-1)^i a_i \omega^i\right] j}{\left[\sum_{\substack{i=0 \\ i\ odd}}^{n} (-1)^i b_i \omega^i\right] + \left[\sum_{\substack{i=0 \\ i\ even}}^{n} (-1)^i b_i \omega^i\right] j}$$

$$\left|T(j\omega)\right| = \sqrt{\frac{\left[\sum_{\substack{i=0 \\ i\ odd}}^{m} (-1)^i a_i \omega^i\right]^2 + \left[\sum_{\substack{i=0 \\ i\ even}}^{m} (-1)^i a_i \omega^i\right]^2}{\left[\sum_{\substack{i=0 \\ i\ odd}}^{n} (-1)^i b_i \omega^i\right]^2 + \left[\sum_{\substack{i=0 \\ i\ even}}^{n} (-1)^i b_i \omega^i\right]^2}}$$

$|T(j\omega)|$ is highly nonlinear in $<a_k>$ and $<b_k>$

Least Squares Approximations of Transfer Functions

$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i}$$

WLOG $b_0 = 1$

$$\left|T(j\omega)\right| = \sqrt{\frac{\left[\sum_{\substack{i=0 \\ i\ \text{odd}}}^{m} (-1)^i a_i \omega^i\right]^2 + \left[\sum_{\substack{i=0 \\ i\ \text{even}}}^{m} (-1)^i a_i \omega^i\right]^2}{\left[\sum_{\substack{i=0 \\ i\ \text{odd}}}^{n} (-1)^i b_i \omega^i\right]^2 + \left[\sum_{\substack{i=0 \\ i\ \text{even}}}^{n} (-1)^i b_i \omega^i\right]^2}}$$

Consider the natural cost function

$$C = \sum_{k=1}^{N} \left(\left|T(j\omega_k)\right| - \tilde{T}(\omega_k) \right)^2$$

$$\left.\begin{array}{c} \dfrac{\partial C}{\partial a_k} \\[2ex] \dfrac{\partial C}{\partial b_k} \end{array}\right\}$$

both are highly nonlinear in $<a_k>$ and $<b_k>$

Closed form solution for optimal values of $<a_k>$ and $<b_k>$ does not exist

Least Squares Approximations of Transfer Functions

$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i} \qquad \text{WLOG } b_0 = 1$$

Consider

$$H_A\left(\omega^2\right) = \frac{\sum_{i=0}^{m} c_i \omega^{2i}}{\sum_{i=0}^{n} d_i \omega^{2i}}$$

Consider the cost function

$$C = \sum_{k=1}^{N}\left(H_A\left(\omega_k^2\right) - \tilde{H}\left(\omega_k^2\right)\right)^2$$

What about the sets of equations $\left\langle \frac{\partial C}{\partial c_k} \right\rangle_{k=1}^{m}$ and $\left\langle \frac{\partial C}{\partial d_k} \right\rangle_{k=1}^{n}$

Rewriting the cost function

$$C = \sum_{k=1}^{N}\left(\frac{\sum_{i=0}^{m} c_i \omega^{2i}}{\sum_{i=0}^{n} d_i \omega^{2i}} - \tilde{H}\left(\omega_k^2\right)\right)^2 \quad \Rightarrow \quad C = \sum_{k=1}^{N}\left(\frac{\sum_{i=0}^{m} c_i \omega^{2i} - \tilde{H}\left(\omega_k^2\right)\sum_{i=0}^{n} d_i \omega^{2i}}{\sum_{i=0}^{n} d_i \omega^{2i}}\right)^2$$

$\left\langle \frac{\partial C}{\partial c_k} \right\rangle_{k=1}^{m}$ is linear in $\langle c_k \rangle$

$\left\langle \frac{\partial C}{\partial d_k} \right\rangle_{k=1}^{n}$ is highly nonlinear in $\langle d_k \rangle$

Closed form solution for optimal values of $\langle c_k \rangle$ and $\langle d_k \rangle$ does not exist

Least Squares Approximations of Transfer Functions

$$H_A\left(\omega^2\right)=\frac{\sum_{i=0}^{m} c_i \omega^{2i}}{\sum_{i=0}^{n} d_i \omega^{2i}}$$

$$C = \sum_{k=1}^{N}\left(H_A\left(\omega_k^2\right)-\tilde{H}\left(\omega_k^2\right)\right)^2$$

$$C = \sum_{k=1}^{N}\left(\frac{\sum_{i=0}^{m} c_i \omega^{2i}-\tilde{H}\left(\omega_k^2\right)\sum_{i=0}^{n} d_i \omega^{2i}}{\sum_{i=0}^{n} d_i \omega^{2i}}\right)^2$$

$\left\langle \frac{\partial C}{\partial c_k} \right\rangle_{k=1}^{m}$ is linear in $<c_k>$ $\left\langle \frac{\partial C}{\partial d_k} \right\rangle_{k=1}^{n}$ is highly nonlinear in $<d_k>$

But

if $<d_k>$ is fixed, optimal value of $<c_k>$ can be easily obtained

equivalently,

if poles of $H_A(\omega^2)$ are fixed, optimal value of zeros of $H_A(\omega^2)$ can be easily obtained

Is this observation useful?

Least Squares Approximations of Transfer Functions

$$C=\sum_{k=1}^{N}\left(\frac{\sum_{i=0}^{m}c_i\omega^{2i}-\tilde{H}\left(\omega_k^2\right)\sum_{i=0}^{n}d_i\omega^{2i}}{\sum_{i=0}^{n}d_i\omega^{2i}}\right)^2$$

if poles of $H_A(\omega^2)$ are fixed, optimal value of zeros of $H_A(\omega^2)$ can be easily obtained

$$C=\sum_{k=1}^{N}\left(\frac{\sum_{i=0}^{m}c_i\omega^{2i}-\tilde{H}\left(\omega_k^2\right)\sum_{i=0}^{n}d_i\omega^{2i}}{\sum_{i=0}^{n}\hat{d}_i\omega^{2i}}\right)^2$$

if poles of $H_A(\omega^2)$ are fixed in denominator of C, the partials of C wrt both $<c_k>$ and $<d_k>$ are linear in $<c_k>$ and $<d_k>$

Are these observations useful?

- Several optimization approaches can be derived from these observations
- Some may provide a LMS optimization of $H_A(\omega^2)$
- No guarantee that inverse mapping exists
- Some may provide a good approximation even though not truly LMS
- Others may not be useful

Least Squares Approximations of Transfer Functions

$$C = \sum_{k=1}^{N} \left(\frac{\sum_{i=0}^{m} c_i \omega^{2i} - \tilde{H}\left(\omega_k^2\right) \sum_{i=0}^{n} d_i \omega^{2i}}{\sum_{i=0}^{n} d_i \omega^{2i}} \right)^2$$

Possible uses of these observations (four algorithms)

1. Guess poles and obtain optimal zero locations
2. Start with a “good” T(s) obtained by any means and improve by selecting optimal zeros
3. Guess poles and then update estimates of both poles and zeros, use new estimate of poles and again update both zeros and poles, continue until convergence or stop after fixed number of iterations
4. Guess poles and obtain optimal zeros. Then invert function and cost and obtain optimal zeros (which are actually poles). Then invert again and obtain optimal zeros. Process can be repeated. - Weighting may be necessary to de-emphasize stop-band values when working with the reciprocal function

Least Squares Approximations of Transfer Functions

$$C=\sum_{k=1}^{N}\left(\frac{\sum_{i=0}^{m}c_i\omega^{2i}-\tilde{H}\left(\omega_k^2\right)\sum_{i=0}^{n}d_i\omega^{2i}}{\sum_{i=0}^{n}d_i\omega^{2i}}\right)^2$$

Comments/Observations about LMS approximations

1. **As with collocation, there is no guarantee that $T_A(s)$ can be obtained from $H_A(\omega^2)$**
2. **Closed-form analytical solutions exist for some useful mean square based cost functions**
3. **Any of the LMS cost functions discussed that have an analytical solution can have the terms weighted by a weight w_i. This weight will not change the functional form of the equations but will affect the fit**
4. **The best choice of sample frequencies is not obvious (both number and location)**
5. **The LMS cost function is not a natural indicator of filter performance**
6. **It is often used because more natural indicators are generally not mathematically tractable**
7. **The LMS approach may provide a good solution for some classes of applications but does not provide a universal solution**

Least Squares Approximations of Transfer Functions

$$C = \sum_{k=1}^{N} \left(\frac{\sum_{i=0}^{m} c_i \omega^{2i} - \tilde{H}\left(\omega_k^2\right) \sum_{i=0}^{n} d_i \omega^{2i}}{\sum_{i=0}^{n} d_i \omega^{2i}} \right)^2$$

Comments/Observations about LMS approximations

1. **As with collocation, there is no guarantee that $T_A(s)$ can be obtained from $H_A(\omega^2)$**
2. **Closed-form analytical solutions exist for some useful mean square based cost functions**
3. **Any of the LMS cost functions discussed that have an analytical solution can have the terms weighted by a weight w_i. This weight will not change the functional form of the equations but will affect the fit**
4. **The best choice of sample frequencies is not obvious (both number and location)**
5. **The LMS cost function is not a natural indicator of filter performance**
6. **It is often used because more natural indicators are generally not mathematically tractable**
7. **The LMS approach may provide a good solution for some classes of applications but does not provide a universal solution**

Least Squares Approximations of Transfer Functions

$$C = \sum_{k=1}^{N}\left(\frac{\sum_{i=0}^{m} c_i \omega^{2i} - \tilde{H}\left(\omega_k^2\right)\sum_{i=0}^{n} d_i \omega^{2i}}{\sum_{i=0}^{n} d_i \omega^{2i}}\right)^2$$

The LMS cost function is not a natural indicator of filter performance

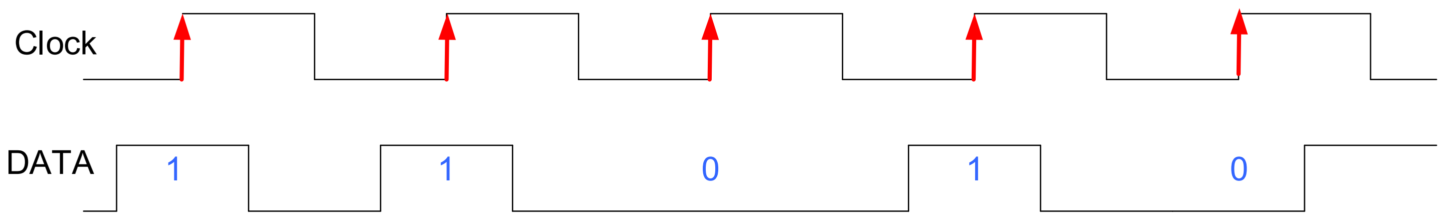
What is a natural indicator of filter performance?

- Strongly dependent upon application
- System designer may specify a filter (e.g. BP) with certain characteristics without knowing what is really needed

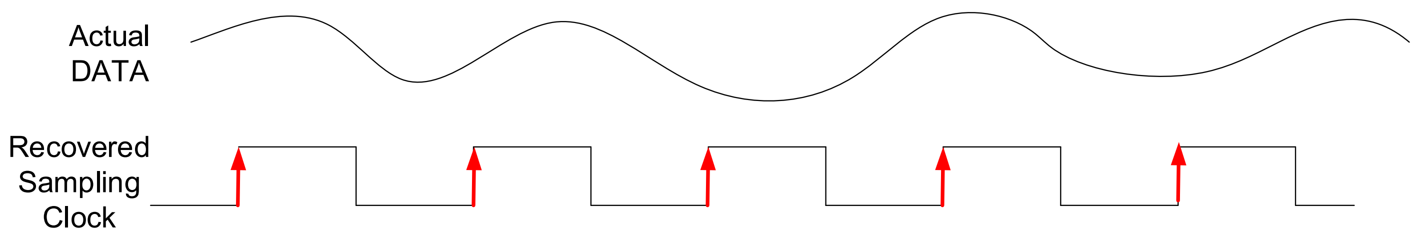
e.g. A filter in a PLL that is used for Clock and Data Recovery may affect capture time, lock time, and BER and those would be the metrics that should be used to determine filter requirements but relationship between pole and zero locations or magnitude or phase response of the filter and these metrics is generally not analytically tractable

Clock and Data Recovery Example

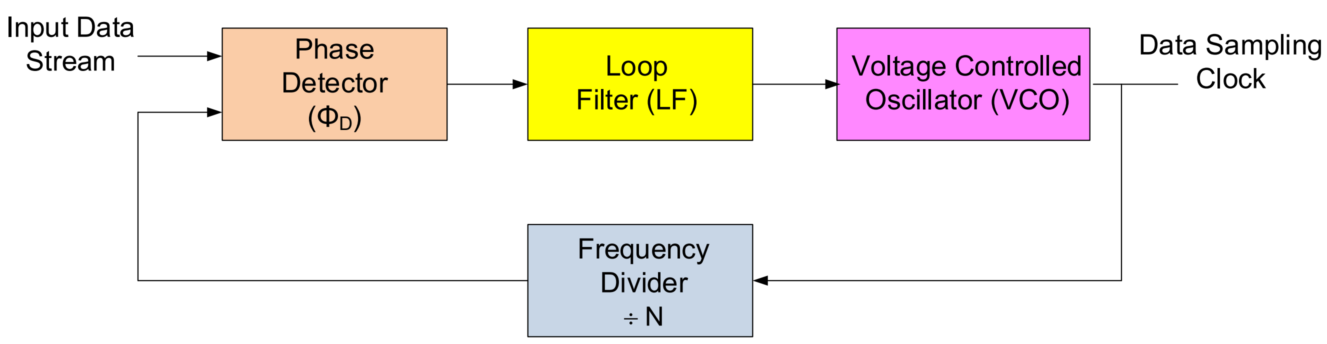
Ideal CDR situation



Actual CDR situation

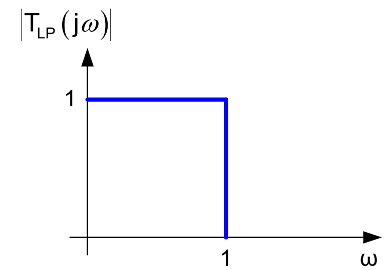


PLL used to generate recovered sampling clock



Relationship between LF transfer function and BER ?

The Approximation Problem



Approach we will follow:

- Magnitude Squared Approximating Functions $H_A(\omega^2)$
- Inverse Transform $H_A(\omega^2) \rightarrow T_A(s)$
- Collocation
- Least Squares (Cost function minimization)
- Pade' Approximations
- Other Analytical Optimization
- Numerical Optimization
- Canonical Approximations
  - $\rightarrow$ Butterworth (BW)
  - $\rightarrow$ Chebyschev (CC)
  - $\rightarrow$ Elliptic
  - $\rightarrow$ Bessel
  - $\rightarrow$ Thomson

Pade' Approximations

Henri Eugène Padé (December 17, 1863 – July 9, 1953) was a French mathematician, who is now remembered mainly for his development of approximation techniques for functions using rational functions.

The Pade' approximations were discussed in his doctoral dissertation in approximately 1890

(from Wikipedia)

Pade' Approximations

Consider the polynomial

$$T_D(s) = \sum_{i=0}^{\infty} c_i s^i$$

Define the rational fraction Rm,n(s) by

$$R_{m,n}(s) = \frac{\sum_{i=0}^{m} a_i s^i}{1 + \sum_{i=1}^{n} b_i s^i} = \frac{A(s)}{B(s)}$$

The rational fraction $R_{m,n}(s)$ is said to be a (m,n)th order Pade' approximation of $T_D(s)$ if $T_D(s)B(s)$ agrees with A(s) through the first m+n+1 powers of s

Note the Pade' approximation applies to any polynomial with the argument being either real, complex, or even an operator s

Can operate directly on functions in the s-domain

Pade' Approximations

Example

$$T_D(s) = 1 + s + \left(\frac{1}{2!}\right)s^2 + \left(\frac{1}{3!}\right)s^3 + \ldots$$

Determine $R_{2,3}(s)$

$$R_{2,3}(s) = \frac{a_0 + a_1 s + a_2 s^2}{1 + b_0 + b_1 s + b_2 s^2 + b_3 s^3} = \frac{A(s)}{B(s)}$$

setting

$$T_D(s)B(s) = A(s)$$

obtain

$$\left(1 + s + \left(\frac{1}{2!}\right)s^2 + \left(\frac{1}{3!}\right)s^3 + \ldots\right)\left(1 + b_1 s + b_2 s^2 + b_3 s^3\right) = a_0 + a_1 s + a_2 s^2$$

Pade' Approximations

Example

$$T_D(s)=1+s+\left(\frac{1}{2!}\right)s^2+\left(\frac{1}{3!}\right)s^3+\ldots$$

$$\left(1+s+\left(\frac{1}{2!}\right)s^2+\left(\frac{1}{3!}\right)s^3+\ldots\right)\left(1+b_1s+b_2s^2+b_3s^3\right)=a_0+a_1s+a_2s^2$$

$$\left.\begin{aligned}
&a_0=1\\
&a_1=1+b_1\\
&a_2=b_1+b_2+\frac{1}{2!}\\
&0=b_2+b_3+\frac{b_1}{2}+\frac{1}{6}\\
&0=b_3+\frac{b_2}{2}+\frac{b_1}{6}+\frac{1}{24}\\
&0=\frac{b_3}{2}+\frac{b_2}{6}+\frac{b_1}{24}+\frac{1}{5!}
\end{aligned}\right\} \Rightarrow$$

$b_1=-.6$

$b_2=.15$

$b_3=-.01666$

$a_0=1$

$a_1=0.4$

$a_2=.05$

Pade' Approximations

Example

b_1=-.6

b_2=.15

b_3=-.01666

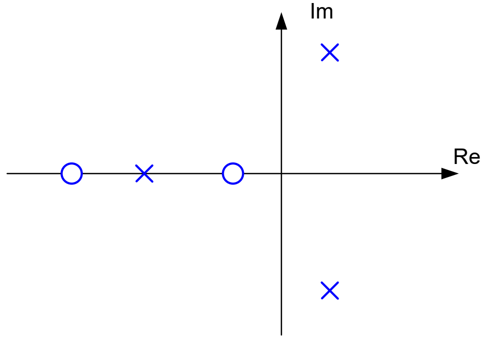
a_0=1

a_1=0.4

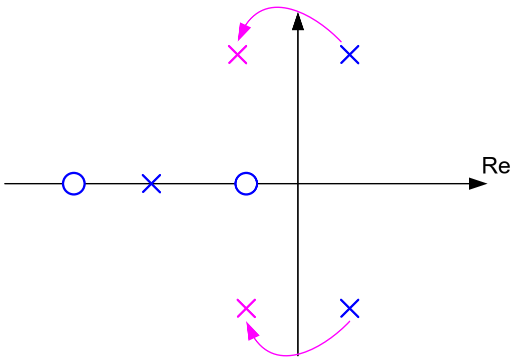
a_2=.05

$$T(s) = \frac{1 + 0.4\,s + 0.05s^2}{1 - 0.6s + 0.15s^2 - 0.01\overline{6}\,s^3}$$

T(s) has a pair of cc poles in the RHP and is thus unstable!



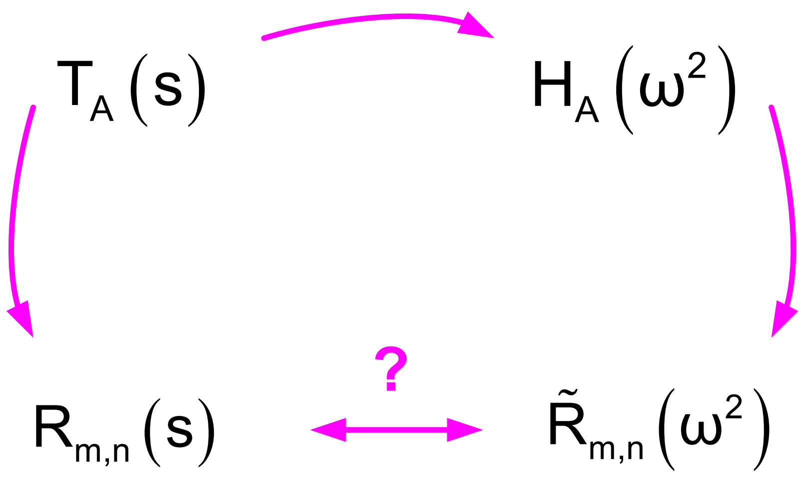
Poles can be reflected back into the LHP to obtain stability and maintain magnitude response



Pade' Approximations

If $T_A(s)$ is an all pole approximation, then the Pade' approximation of $1/T_A(s)$ is the reciprocal of the Pade' approximation of $T_A(s)$

Pade' approximations can be made for either $T_A(s)$ or $H_A(\omega^2)$.



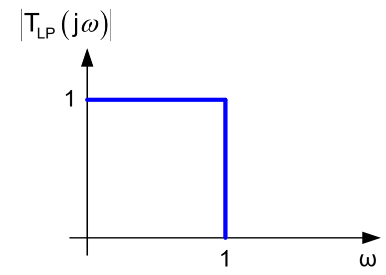
Is it better to do Pade' approximations of $T_A(s)$ or $H_A(\omega^2)$?

What relationship, if any, exists between $R_{m,n}(s)$ and $\tilde{R}_{m,n}(s)$?

Pade’ Approximations

- Useful for order reduction of all-pole or all-zero approximations
- Can map an all-zero approximation to a realizable rational fraction in the s-domain
- Can extend concept to provide order reduction of higher-order rational fraction approximations
- Can always maintain stability or even minimum phase by reflecting any RHP roots back into the LHP
- Pade’ approximation is heuristic (no metrics associated with the approach)
- No guarantees about how good the approximations will be

The Approximation Problem



Approach we will follow:

- Magnitude Squared Approximating Functions $H_A(\omega^2)$
- Inverse Transform $H_A(\omega^2) \rightarrow T_A(s)$
- Collocation
- Least Squares (Cost function minimization)
- Pade' Approximations
- Other Analytical Optimization
- Numerical Optimization
- Canonical Approximations
  - → Butterworth (BW)
  - → Chebyschev (CC)
  - → Elliptic
  - → Bessel
  - → Thomson

Other Analytical Approximations

- Numerous analytical strategies have been proposed over the years for realizing a filter
- Some focus on other characteristics (phase, time-domain response, group delay)
- Almost all based upon real function approximations
- Remember – inverse mapping must exist if a useful function T(s) is to be obtained

Approximations

- Magnitude Squared Approximating Functions – $H_A(\omega^2)$
- Inverse Transform - $H_A(\omega^2) \rightarrow T_A(s)$
- Collocation
- Least Squares (Cost function minimization)
- Pade Approximations
- Other Analytical Optimizations
- Numerical Optimization
- Canonical Approximations
  - Butterworth
  - Chebyschev
  - Elliptic
  - Bessel
  - Thomson

Numerical Optimization

- Optimization algorithms can be used to obtain approximations in either the s-domain or the real domain
- The optimization problem often has a large number of degrees of freedom (m+n+1)

$$T(s)=\frac{\sum_{k=0}^{m} a_k s^k}{1+\sum_{k=0}^{n} b_k s^k}$$

- Need a good cost function to obtain good approximation
- Can work on either coefficient domain or root domain or other domains
- Rational fraction approximations inherently vulnerable to local minimums
- Can get very good results

Stay Safe and Stay Healthy !

End of Lecture 8